

# Thermal dimension of quantum spacetime

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## Abstract

Recent results suggest that a crucial crossroad for quantum gravity is the characterization of the effective dimension of spacetime at short distances, where quantum properties of spacetime become significant. This is relevant in particular for various scenarios of “dynamical dimensional reduction” which have been discussed in the literature. We are here concerned with the fact that the related research effort has been based mostly on analyses of the “spectral dimension”, which involves an unphysical Euclideanization of spacetime and is highly sensitive to the off-shell properties of a theory. As here shown, different formulations of the same physical theory can have wildly different spectral dimension. We propose that dynamical dimensional reduction should be described in terms of the “thermal dimension” which we here introduce, a notion that only depends on the physical content of the theory. We analyze a few models with dynamical reduction both of the spectral dimension and of our thermal dimension, finding in particular some cases where thermal and spectral dimension agree, but also some cases where the spectral dimension has puzzling properties while the thermal dimension gives a different and meaningful picture.

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## 1. Introduction.

There are several alternative approaches to the study of the quantum-gravity problem, with formalizations and physical pictures that are significantly different, in most cases offering very few opportunities to compare predictions between one approach and another. As a result, there is strong interest for the few features which are found to arise in several alternative models. A common expectation is that at short distances the classical picture of spacetime as a Riemannian geometry should be replaced by some new “quantum” geometry. The alternative pictures of quantum spacetime appear to be rather different, but over the last decade it became clear that some of these have in common the mechanism of “dynamical dimensional reduction”: the familiar four-dimensional classical picture of spacetime in the IR (“infrared”, *i.e.* for probes of wavelength much longer than the Planck length) is replaced by a quantum picture with an effective number of spacetime dimensions smaller than four in the UV (“ultraviolet”, *i.e.* for probes of wavelength comparable to the Planck length). These exciting recent developments face the challenge that the standard concept of dimension of a spacetime, the “Hausdorff dimension”, is inapplicable to a quantum spacetime [1, 2], and therefore one must rely on some suitable new concept. This challenge has been handled so far mostly<sup>1</sup> by resorting to the notion of “spectral dimension”, whose key

ingredient is the (modified) d’Alembertian of the theory<sup>2</sup> and for classical flat spacetimes reproduces the Hausdorff dimension. It was in terms of the spectral dimension that dynamical dimensional reduction was described for several approaches to the quantum-gravity problem, including the approach based on Causal Dynamical Triangulations [8], the Asymptotic-Safety approach [9], Horava-Lifshitz gravity [10], the Causal-Sets approach [11], Loop Quantum Gravity [12, 13], Spacetime Noncommutativity [14] and theories with Planck-scale curvature of momentum space [15, 16].

It is for us cause of concern that so much of our intuition about the quantum-gravity realm is being attached to analyses based on the spectral dimension, which is not a physical characterization of a theory. For such precious cases where a feature is found in many approaches to the quantum-gravity problem, and therefore might be a “true feature” of the quantum-gravity realm, we should ask for no less than a fully physical characterization. It is well known that the spectral dimension provides a valuable characterization of properties of classical Riemannian geometries [14, 17], but its proposed applicability to the description of the dimension of a quantum spacetime involves some adaptations, and, as we shall here see, these adaptations are responsible for some of its inadequacies. In the study of quantum spacetimes the spectral dimension is the

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<sup>1</sup>Other candidates for the characterization of the dimension of a quantum spacetime have been proposed in Refs. [2, 3, 4, 5, 6].

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<sup>2</sup>There are cases, such as in Causal Dynamical Triangulations, where the d’Alembertian of the theory is not known, but it is possible to calculate the spectral dimension with other techniques. It has been established [7] that in these cases it is then possible to reconstruct the d’Alembertian.

effective dimension ‘seen’ by a fictitious diffusion process governed by the Euclidean version of the d’Alembertian. The UV value of the spectral dimension,  $d_S(0)$ , is then formally computed via:

$$d_S(0) \equiv -2 \lim_{s \rightarrow 0} \frac{d \ln P(s)}{d \ln(s)}, \quad (1)$$

where  $P(s)$  is the average return probability of the diffusion process and  $s$  is the fictitious diffusion time. When the IR Hausdorff dimension of spacetime is  $D + 1$ , and the Euclidean d’Alembertian of the theory is represented on momentum space as  $\Omega^E(E, p)$ , the return probability is given by<sup>3</sup>

$$P(s) \propto \int dE dp p^{D-1} e^{-s \Omega^E(E, p)}. \quad (2)$$

The fact that the Euclidean version of the d’Alembertian intervenes is of course cause of concern for us<sup>4</sup>. It is in fact well known that the Euclidean version of a quantum-gravity model can be profoundly different from the original model in Lorentzian spacetime (see, *e.g.*, Ref.[20]). Moreover, evidently in (2) an important role is played by off-shell modes, a role so important that, as we shall here show, one can obtain wildly different values for the spectral dimension for different formulations of the same physical theory (cases where the formulations coincide on-shell but are different off-shell). We are also concerned by the fact that evidently the  $P(s)$  of (2) is invariant under active diffeomorphisms on momentum space (an active diffeomorphism on momentum space amounts to an irrelevant change of integration variable for  $P(s)$ ). Since an active diffeomorphism can map a given physical theory into a very different one (also see here below), we believe that this degeneracy of the spectral dimension is worrisome.

While these concerns are, in our appreciation, very serious, we do acknowledge that several analyses centered on the spectral dimension give rather meaningful results. Therefore we are here guided by the idea that it is necessary to replace the spectral dimension with some other fully physical notion of dimensionality of a quantum spacetime, with the requirement that in most cases the new notion should agree with the spectral dimension. Only when the unphysical content of the spectral dimension plays a particularly significant role should the new notion differ significantly from the spectral dimension. In searching for such a new notion we took as guidance the observation

<sup>3</sup> Our thesis here is that even if (2) did describe the return probability (as usually assumed) still the spectral dimension would be unsatisfactory. It is interesting however that, as stressed in Ref.[18], the interpretation of (2) as return probability is not always applicable.

<sup>4</sup>Concerns for the Euclideanization involved were also raised in Ref.[19], within a study concerning the causal-set approach. Ref.[19] proposed a possible redefinition of the spectral dimension suitable for including Lorentzian signature and found that it gave different results with respect to the standard (Euclideanized) spectral dimension.

reported in recent studies [21, 22, 23] (see also [24] for earlier related proposals) that in some instances the Stefan-Boltzmann law gives indications on the dimensionality of spacetime that are consistent with the spectral dimension. One can view the Stefan-Boltzmann law as an indicator of spacetime dimensionality since for a gas of radiation in a classical spacetime with  $D + 1$  dimensions the Stefan-Boltzmann law takes the form

$$U \propto T^{D+1}. \quad (3)$$

Actually several thermodynamical relations are sensitive to the dimensionality of spacetime, another example being the equation of state parameter  $w \equiv P/\rho$ , relating pressure  $P$  and energy density  $\rho$ , which for radiation in a classical spacetime with  $D + 1$  dimensions takes the form

$$w = \frac{1}{D}. \quad (4)$$

These observations inspire our proposal of assigning a “thermal dimension” to a quantum spacetime. Our recipe involves studying the thermodynamical properties of radiation with on-shellness characterized by the (deformed) d’Alembertian of the relevant quantum-spacetime theory (the same deformed d’Alembertian used when evaluating the spectral dimension, but in its Lorentzian form). By looking at the resulting Stefan-Boltzmann law and equation of state one can infer the effective dimensionality of the relevant quantum spacetime. This notion of dimensionality has the advantage of being directly observable, a genuine physical property of the quantum spacetime, and, as we shall here show, fixes the shortcomings of the spectral dimension, while agreeing with it in some particularly noteworthy cases.

## 2. Application to generalized Horava-Lifshitz scenarios.

We start the quantitative part of our study by considering a class of generalized Horava-Lifshitz scenarios, which has been the most active area of research on dynamical dimensional reduction [10, 7, 15]. These are cases where the momentum-space representation of the deformed d’Alembertian takes the form

$$\Omega_{\gamma_t \gamma_x}(E, p) = E^2 - p^2 + \ell_t^{2\gamma_t} E^{2(1+\gamma_t)} - \ell_x^{2\gamma_x} p^{2(1+\gamma_x)}. \quad (5)$$

where  $E$  is the energy,  $p$  is the modulus of the spatial momentum,  $\gamma_t$  and  $\gamma_x$  are dimensionless parameters, and  $\ell_t$  and  $\ell_x$  are parameters with dimension of length (usually assumed to be of the order of the Planck length).

For this model it is known [15, 7] that the UV value of the spectral dimension, obtained from the Euclidean version of the above d’Alembertian ( $E^2 + p^2 + \ell_t^{2\gamma_t} E^{2(1+\gamma_t)} + \ell_x^{2\gamma_x} p^{2(1+\gamma_x)}$ ), is

$$d_S(0) = \frac{1}{1 + \gamma_t} + \frac{D}{1 + \gamma_x}. \quad (6)$$

In deriving the thermal dimension for this case we start from the logarithm of the thermodynamical partition function [25], written so that the integration is explicitly taken over the full energy-momentum space:

$$\log Q_{\gamma_t \gamma_x} = -\frac{2V}{(2\pi)^3} \int dE d^3p \left[ \delta(\Omega_{\gamma_t \gamma_x}) \Theta(E) \cdot \right. \\ \left. \cdot 2E \log(1 - e^{-\beta E}) \right]. \quad (7)$$

Here  $\beta$  is related to the Boltzmann constant  $k_B$  and temperature via  $\beta = \frac{1}{k_B T}$ , and the delta function  $\delta(\Omega_{\gamma_t \gamma_x})$  enforces the on-shell relation  $\Omega_{\gamma_t \gamma_x} = 0$ .

From (7) one obtains the energy density and the pressure respectively as

$$\rho_{\gamma_t \gamma_x} \equiv -\frac{1}{V} \frac{\partial}{\partial \beta} \log Q_{\gamma_t \gamma_x}, \quad p_{\gamma_t \gamma_x} \equiv \frac{1}{\beta} \frac{\partial}{\partial V} \log Q_{\gamma_t \gamma_x}. \quad (8)$$

In Figure 1 we show (for a few choices of  $\gamma_x, \gamma_t$ ) the resulting temperature dependence for the energy density and for the equation of state parameter. For the UV/high-temperature values of  $\rho_{\gamma_t \gamma_x}$  and  $w_{\gamma_t \gamma_x}$  one can easily establish the following behaviors at high temperature, in agreement with the content of Figure 1

$$\rho_{\gamma_t \gamma_x} \propto T^{1+3\frac{1+\gamma_t}{1+\gamma_x}}, \quad w_{\gamma_t \gamma_x} = \frac{1+\gamma_x}{3(1+\gamma_t)}. \quad (9)$$

By comparison to (3) and (4) one sees that both of these results give a consistent prediction for the “thermal dimension” at high temperature, which is

$$d_T = 1 + 3\frac{1+\gamma_t}{1+\gamma_x}. \quad (10)$$

Interestingly, in this case of generalized Horava-Lifshitz scenarios the thermal dimension agrees with spectral dimension, eq. (6), for  $\gamma_t = 0$ , but differs from the spectral dimension when  $\gamma_t \neq 0$ .

### 3. Implications of active diffeomorphisms on momentum space.

Generalized Horava-Lifshitz scenarios also give us an easy opportunity for comparing the properties of the thermal dimension and of the spectral dimension under active diffeomorphisms on momentum space. From this perspective the analysis is particularly simple for the case  $\gamma_x = 0, \gamma_t = 1$ , where one has

$$\Omega_{1,0}(E, p) = E^2 - p^2 + \ell_t^2 E^4. \quad (11)$$

In light of the results reviewed and derived above we know that in this case the UV spectral dimension is  $d_S = 3.5$ , while the UV thermal dimension is  $d_T = 7$ .

Let us then contemplate a simple diffeomorphism on momentum space, the following reparameterization of the energy variable:  $E \rightarrow \tilde{E} = \sqrt{E^2 + \ell_t^2 E^4}$ . In terms of  $\tilde{E}$  the d’Alembertian takes the standard special-relativistic

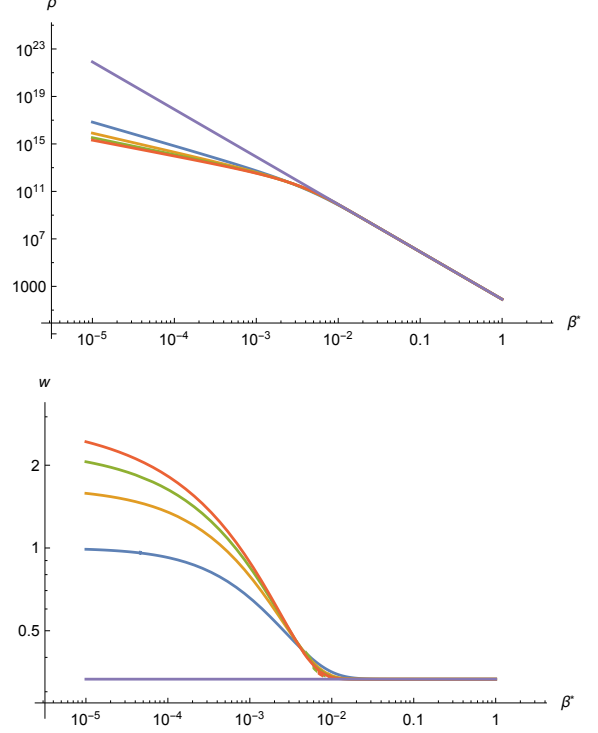


Figure 1: Behavior of the energy density  $\rho$  in arbitrary units (top panel) and of the equation of state parameter  $w$  (bottom panel) as a function of  $\beta^* \equiv 10^{-3} \beta k_B T_P$ , according to the partition function  $Q_{\gamma_t \gamma_x}$ , for  $\gamma_t = 0$  and  $\gamma_x = 2$  (blue),  $\gamma_x = 4$  (orange),  $\gamma_x = 6$  (green),  $\gamma_x = 8$  (red). The purple line is the standard case,  $\rho \propto T^4$  (top panel) and  $w = 1/3$  (bottom panel).

form,  $\Omega_{1,0} = \tilde{E}^2 - p^2$ , while the momentum space measure becomes non-trivial:

$$d\mu(\tilde{E}, p) = \frac{d\tilde{E} dp \sqrt{2} \ell_t p^2 \tilde{E}}{\sqrt{(1 + 4\ell_t^2 \tilde{E}^2)(-1 + \sqrt{1 + 4\ell_t^2 \tilde{E}^2})}} \quad (12)$$

When the above diffeomorphism on momentum space is an active one, the laws of physics are not invariant. This is indeed what is found when comparing the thermodynamical properties of the “ $\tilde{E}, p$  theory” with d’Alembertian  $\tilde{E}^2 - p^2$  and momentum-space integration measure (12) and the “ $E, p$  theory” with (deformed) d’Alembertian  $\Omega_{1,0}(E, p) = E^2 - p^2 + \ell_t^2 E^4$  and integration measure  $dE d^3p$ . In the “ $\tilde{E}, p$  theory” the logarithm of the thermodynamical partition function is

$$\log \tilde{Q}_{act.} = -\frac{2V}{(2\pi)^3} \int d\mu(\tilde{E}, p) \left[ \delta(\tilde{E}^2 - p^2) \Theta(\tilde{E}) \cdot \right. \\ \left. \cdot 2\tilde{E} \log(1 - e^{-\beta \tilde{E}}) \right] \neq \log Q. \quad (13)$$

Of course ultimately this leads us to obtain different values for the thermal dimension of these two theories. In fact, from the partition function (13) one can easily find that at high temperatures the energy density behaves as  $\rho \sim T^{3.5}$ , while the equation of state parameter is  $w = 0.4$ . These numbers point at a value of the UV thermal dimension of

$d_T = 3.5$ . Note that this result is different from the one that would follow from a passive diffeomorphism. In this case, the partition function in the  $\tilde{E}, p$  variables would be straightforwardly obtained by a change of variables in eq. (7):

$$\begin{aligned} \log \tilde{Q}_{pass.} &= -\frac{2V}{(2\pi)^3} \int d\mu(\tilde{E}, p) \left[ \delta(\tilde{E}^2 - p^2) \cdot \right. \\ &\quad \left. \Theta(E(\tilde{E})) 2E(\tilde{E}) \log \left( 1 - e^{-\beta E(\tilde{E})} \right) \right] \\ &= \log Q. \end{aligned} \quad (14)$$

A passive diffeomorphism just relabels the same physical picture and of course the thermal dimension is not affected.

On the other hand, it can be easily seen that the spectral dimension is not only invariant under passive diffeomorphisms but also under active diffeomorphisms on momentum space. In fact, active and passive diffeomorphisms have the same effect on the return probability  $P(s)$  (eq. (2)), that of changing the integration variable (without changing the integral). Therefore the “ $\tilde{E}, p$  theory” has the same UV spectral dimension ( $d_S = 3.5$ ) as the “ $E, p$  theory”.

In summary, one finds that the UV spectral dimension of both the “ $\tilde{E}, p$  theory” and the “ $E, p$  theory” is 3.5, and 3.5 is also the value of the thermal dimension of the “ $\tilde{E}, p$  theory”, but the “ $E, p$  theory” has UV thermal dimension of 7. It should be evidently seen as advantageous for the thermal dimension<sup>5</sup> the fact that it assigns different UV dimension to the two very different “ $E, p$  theory” and “ $\tilde{E}, p$  theory”.

#### 4. Application to $f(E^2 - p^2)$ scenarios.

Another scenario of significant interest is the one where the d'Alembertian is deformed into a function of itself:  $E^2 - p^2 \rightarrow f(E^2 - p^2)$ . The structure of this scenario is very valuable for our purposes, but it also has intrinsic interest since it has been proposed on the basis of studies of the Asymptotic-Safety approach [28] and of the approach based on Causal Sets [29]. We focus on a case which might deserve special interest from the quantum-gravity perspective, as stressed in Ref.[28], such that the deformed d'Alembertian takes the form

$$\Omega_\gamma(E, p) = E^2 - p^2 - \ell^{2\gamma} (E^2 - p^2)^{1+\gamma}, \quad (15)$$

where the parameter  $\gamma$  takes integer positive values and  $\ell$  is a parameter with dimension of length.

<sup>5</sup>Some of us had contemplated in previous work [15, 26, 27] the possibility of describing the dimension of a quantum spacetime in terms of the duality with momentum space, by resorting to the “Hausdorff dimension of momentum space”. However, at least as formulated in [15, 26, 27], that notion is only applicable to theories of the type of the “ $\tilde{E}, p$  theory”, *i.e.* with undeformed d'Alembertian (but possibly deformed measure of integration on momentum space).

For this case one easily finds<sup>6</sup> that the UV spectral dimension is

$$d_S(0) = \frac{4}{1+\gamma}, \quad (16)$$

but the fact that this notion of the UV dimensionality of spacetime depends on  $\gamma$  is puzzling and points very clearly to the type of inadequacies of the spectral dimension that we are here concerned with. In fact, in the UV limit the parameter  $\gamma$  has no implications for the on-shell/physical properties of the (massless) theory. In general, massless particles governed by  $\Omega_\gamma$  will be on-shell only either when

$$E^2 = p^2$$

or when

$$E^2 = p^2 + \frac{1}{\ell^2},$$

independently of the value of  $\gamma$ . At low energies only  $E^2 = p^2$  is viable. For energies such that  $E \geq 1/\ell$  also the second possibility,  $E^2 = p^2 + \frac{1}{\ell^2}$ , becomes viable. However, in the UV limit the two possibilities become indistinguishable, all particles are governed by  $E \simeq p$  just like in any 4-dimensional spacetime, because as  $E \rightarrow \infty$  one has that  $p^2 + \frac{1}{\ell^2} \simeq p^2$ . So without any need to resort to complicated analyses we know that this theory in the UV limit must behave like a 4-dimensional theory, in contradiction with the mentioned result for the UV spectral dimension.

The UV value of our “thermal dimension” is correctly 4, independently of  $\gamma$ . This is easily seen by taking into account the deformation of d'Alembertian present in the  $\Omega_\gamma$  of (15) for the analysis of the partition function:

$$\log Q_\gamma = -\frac{2V}{(2\pi)^3} \int dE d^3p \delta(\Omega_\gamma) \Theta(E) 2E \log(1 - e^{-\beta E}), \quad (17)$$

Using the fact that

$$\delta(\Omega_\gamma) = \frac{\delta(E - p)}{2p} + \frac{\delta(E - \sqrt{p^2 + \frac{1}{\ell^2}})}{2\gamma \sqrt{p^2 + \frac{1}{\ell^2}}}. \quad (18)$$

one easily finds that the UV behavior of thermodynamical quantities which is relevant to determine the thermal dimension is independent of  $\gamma$ , and in particular in the UV the Stefan-Boltzmann law and the equation-of-state parameter take the form known for a standard 4-dimensional spacetime:

$$\rho \propto T^4, \quad w = \frac{1}{3}. \quad (19)$$

So indeed in this scenario the UV value of the thermal dimension is 4. The theory does have “dynamical running of the dimensionality of spacetime” in a regime where the temperature is close to the Planckian temperature, as one should expect on the basis of the fact that the parameter  $\gamma$  does have a role in the theory when the energy is greater

<sup>6</sup>This elementary computation is reported in Ref. [30].

than  $1/\ell$ , as long as the energy is not big enough to render  $p^2$  and  $p^2 + \frac{1}{\ell^2}$  indistinguishable. This is shown in figure 2, where we plot the thermal dimension (inferred from the behaviour of the equation of state parameter and from the running of the energy density with temperature) as a function of  $\beta$ .

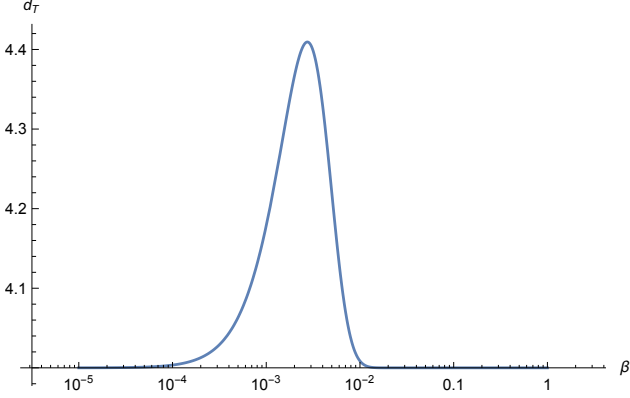


Figure 2: Behaviour of the thermal dimension  $d_T$  as a function of  $\beta$ . The thermal dimension is computed as  $d_T = 1 + \frac{1}{w}$ , where the equation of state parameter  $w$  is the one associated to the partition function  $\log Q_\gamma$ , with  $\gamma = 1$ .  $\beta$  is in units of  $10^3 \beta_P$  (where  $\beta_P = \frac{1}{k_B T_P}$  and  $T_P$  is the Planck temperature).

We attribute the disastrous failures of the spectral dimension in this case to a combination of its sensitivity to off-shell properties and its reliance on the Euclidean d'Alembertian. It is noteworthy that for the Euclidean d'Alembertian<sup>7</sup>,

$$\Omega_\gamma^{[Euclidean]} = E^2 + p^2 + \ell^{2\gamma}(E^2 + p^2)^{1+\gamma}, \quad (20)$$

in the UV limit one can neglect  $E^2 + p^2$  with respect to  $\ell^{2\gamma}(E^2 + p^2)^{1+\gamma}$ . Instead for on-shell modes of the original Lorentzian  $\Omega_\gamma$  one can never neglect  $E^2 - p^2$  with respect to  $\ell^{2\gamma}(E^2 - p^2)^{1+\gamma}$ .

## 5. Closing remarks.

The exciting realization that the UV dimension of spacetime might be different from its IR dimension adds significance to the old challenge of describing the dimension of a quantum spacetime. We here argued that it is crucial to link this issue to observable properties. After all what we mean in physics by “dimension of spacetime” must inevitably be something we measure. Moreover, only by relying on a truly physical/observable characterization are we assured to compare different theories in conclusive manner.

We here exposed fully the inadequacy of the spectral dimension for these purposes. The fact that this notion involves an unphysical Euclideanization could already lead

to this conclusion. We feel that our observation about the undesirable invariance of the spectral dimension under active diffeomorphisms of momentum space should cast another shadow on the usefulness of the spectral dimension. The fact that one obtains different spectral dimensions for alternative formulations of the same physical theory (formulations that differ only for what concerns unphysical off-shell modes) should leave no residual doubts.

We here proposed a notion of dimensionality which is free from the shortcomings of the spectral dimension, since it relies on the analysis of observable thermodynamical properties of radiation in the quantum spacetime. Only experience with its use will gradually say if our notion of thermal dimension of a quantum spacetime is not only physical but also particularly useful. We conjecture that it will prove to be very valuable at least for studies of the early universe, which is anyway the context where the UV dimension of spacetime should find its most significant applications [32, 33].

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<sup>7</sup>Note that in order to have the Euclidean version of the d'Alembertian  $\Omega_\gamma(E, p)$  one has to Wick-rotate also the parameter  $\ell$  [31].

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